IS A MOVING MASS RETARDED BY THE REACTION OF ITS OWN RADIATION?

BY LEIGH PAGE.

SINCE the promulgation of the principle of relativity by Einstein in 1905, a number of alleged inconsistencies with the classical theory of electrodynamics have been pointed out. That these apparent inconsistencies must be due to failure to analyze correctly the problem under consideration, and that the electrodynamic equations can in no way come into contradiction with the principle of relativity—reference here is to the relativity of constant velocity systems, not to the broader conception of general relativity recently developed by Einstein—might have been surmised from the very first, for Lorentz[1] had already shown that the electrodynamics of moving systems could be reduced to that of fixed systems by a group of transformations substantially the same as those deduced by Einstein from the principle of relativity. Moreover, looking at the question from the other side, the author[2] of this paper has shown that the electrodynamic equations may be obtained in their entirety and exactly, from nothing more than the kinematical transformations of relativity and the assumption that each and every element of charge is a center of uniformly diverging tubes of strain. Hence, although the electrodynamic equations may not cover as broad a ground as the principle of relativity, they can contain nothing that is in contradiction with this principle.

One of the most important supposed inconsistencies of the principle of relativity with classical electrodynamics has been connected with the phenomenon of anomalous dispersion. Here we have an index of refraction less than unity, leading, apparently, to the conclusion that the velocity of light in the dispersing medium is greater than the velocity of light in vacuo. Since the essence of the kinematics of relativity lies in the fact that the velocity of light in vacuo shall be an absolute maximum, it seemed at first sight that here we had an experimental disproof of the conception of relativity. Not until the masterly papers of Sommerfeld and Brillouin[3] were published in 1914 was the matter finally cleared up.

[1] Theory of Electrons, p. 197.

[2] "Relativity and the Ether," Am. Jour. of Sci., 38, p. 169, 1914.

[3] Ann. d. Physik, 44, p. 177, 1914.

These authors showed that the velocity with which the index of refraction is concerned is a "phase" velocity, and not a "signal" velocity. By a very ingenious mathematical method they were able to investigate the propagation of a wave train of limited length through a material medium, whether in the region of anomalous dispersion or not, and to show that the velocity of the front of the disturbance, *i. e.*, the "forerunners," would be always exactly the same as the velocity of light in vacuo—never greater, never less.

Another criticism of the principle of relativity of the same nature as the above, although not concerned with electrodynamics, is based on the alleged possibility of transmitting a signal with a velocity greater than the velocity of light by means of a gravitational disturbance. More than one author refers to the "immense . . . speed of propagation of gravitation," [1] although it has repeatedly been pointed out that none of the facts revealed by astronomical investigation requires for its explanation a velocity of propagation for gravitation greater than the velocity of light.[2]

The object of the present paper is to clear up what is, so far as the author is aware, the only supposed inconsistency of the principle of relativity with classical electrodynamics which remains a subject of serious consideration on the part of contemporaneous physicists. This is the radiation reaction experienced by a moving mass on account of its own emission of radiant energy. The problem is treated in some detail by Professor Sir Joseph Larmor in the Proceedings of the Fifth International Congress of Mathematicians[2] held at Cambridge in 1912, and in a recent number of Nature[3] he emphasizes the contradiction to the principle of relativity involved in his solution of this problem.

Consider a radiating mass, such as a star, which is moving in a straight line with velocity V. The reaction of its radiation is found by Larmor to constitute a resistance to the velocity equal to

$$\mathbf{F} = -\frac{1}{c^2} R\mathbf{V}, \tag{1}$$

where c is the velocity of light in vacuo, and R the total energy emitted per unit time.

Now consider an observer A at rest, and a star at rest. The star will remain at rest indefinitely in so far as the reaction of its own radiation is concerned. However the case is quite different if we consider an

[1] Proc. of Fifth International Congress of Math., I., p. 207, 1912.

[2] O. Heaviside, Electromagnetic Theory, I., Appendix *B*; H. A. Lorentz, Amsterdam Proceedings, 2, p. 573, 1900.

[3] Nature, 99, p. 404, 1917.

observer B who is moving in the Z direction with a constant velocity V. A star initially at rest relative to him will gradually acquire a velocity (relative to observer B, of course) in the $-Z$ direction on account of the reaction of its own radiation. This velocity will increase asymptotically until it reaches the final constant value V. Hence the systems of observers A and B cannot be equivalent, and the principle of relativity comes into contradiction with classical electrodynamics when applied to this particular problem.

Such would be the only possible conclusion if the deduction of equation (1) from the electrodynamic equations were correct. In order to point out the tacit assumption which invalidates Larmor's derivation of (1), we shall reproduce in somewhat more rigorous form what is substantially the analytical reasoning pursued by him. Then we shall investigate the problem quite rigorously by a somewhat different method, and show that the electrodynamic equations do *not* lead to a radiation reaction which depends upon the velocity, but to a reaction which is exactly in accord with the principle of relativity. Incidentally we shall develop the complete dynamical equation of an electron to the fifth order.

If we use the units of electric charge and magnetic pole advocated by Heaviside and Lorentz—a unit $1/\sqrt{4\pi}$ smaller than the electrostatic or electromagnetic units respectively—classical electrodynamic theory is contained in the five vector equations[1]

$$\nabla \cdot \mathbf{E} = \rho, \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\dot{\mathbf{H}}, \tag{3}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{4}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}(\dot{\mathbf{E}} + \rho\mathbf{v}), \tag{5}$$

$$\mathbf{F} = \rho\left[\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}\right], \tag{6}$$

where equations (2) to (5) inclusive describe the effect of the distribution of matter upon ether, and (6) gives the effect of ether on matter. From (3), (5), and (6) we obtain at once the familiar energy equation for the region inside the closed surface Σ, namely

$$\frac{d}{dt}\left[\tfrac{1}{2}\int (E^2 + H^2)d\tau\right] + c\int (\mathbf{E} \times \mathbf{H})\cdot d\boldsymbol{\sigma} + \int \mathbf{F}\cdot\mathbf{v}d\tau = 0, \tag{7}$$

where $d\tau$ is an element of volume and $d\boldsymbol{\sigma}$ a vector element of surface

[1] Gibbs's vector notation is used.

having the direction of the outward drawn normal, the volume integrals being taken throughout the volume enclosed by the surface Σ and the surface integral over this surface. The first term represents the rate of increase of electromagnetic energy, the second the rate of escape of energy through the enclosing surface, and the third the rate at which work is done by the field on the matter contained in this region.

Now we are interested in the reaction of the ether on the material oscillators which constitute the radiating body under consideration. To find this reaction we may proceed by either of two equivalent methods, which we shall designate as methods A and B.

METHOD A.

We may eliminate ρ and $\rho\mathbf{v}$ from (6) by means of the field equations (2) to (5). This yields for the resultant force on the matter within the closed surface Σ the familiar expression

$$\mathbf{K} = \int \mathbf{F} d\tau = \int (\mathbf{EE} + \mathbf{HH}) \cdot d\sigma - \tfrac{1}{2} \int (E^2 + H^2) d\sigma - \frac{1}{c}\frac{d}{dt} \int (\mathbf{E} \times \mathbf{H}) d\tau,$$

where the surface integrals are taken over the surface Σ and the volume integral throughout the region enclosed by this surface.

Let us write

$$\mathbf{K}_1 = \int (\mathbf{EE} + \mathbf{HH}) \cdot d\sigma - \tfrac{1}{2} \int (E^2 + H^2) d\sigma, \tag{8}$$

$$\mathbf{K}_2 = -\frac{1}{c}\frac{d}{dt} \int (\mathbf{E} \times \mathbf{H}) d\tau. \tag{9}$$

Then $\mathbf{K}_1$ is the stress which Maxwell considered to be exerted by the ether without the surface Σ on the ether within this surface, and $\mathbf{K}_2$ has been interpreted as the rate of decrease of electromagnetic momentum within the enclosing envelope,

$$\frac{1}{c} (\mathbf{E} \times \mathbf{H})$$

being the momentum of the ether per unit volume.

If, now, we imagine a closed surface to surround the matter on which we wish to find the force $\mathbf{K}$, our problem reduces to the evaluation of the integral expressions for $\mathbf{K}_1$ and $\mathbf{K}_2$. To determine the values of the integrands, however, it is necessary to know the distribution of ρ and $\rho\mathbf{v}$ in space and time, so as to solve the field equations (2) to (5) for $\mathbf{E}$ and $\mathbf{H}$.

METHOD B.

We may solve (2) to (5) for $\mathbf{E}$ and $\mathbf{H}$ in terms of ρ and $\rho\mathbf{v}$, substitute in (6), and evaluate the integral

$$\mathbf{K} = \int \mathbf{F} d\tau, \tag{10}$$

where the volume integral need be taken only over those regions where ρ is not zero, *i. e.*, over the matter on which we wish to find the force **K**.

The second method is somewhat the more direct, and has the great advantage that in most cases the integration covers a very small region, so that if it is necessary to expand **E** and **H** in terms of the distance between the elements of charge considered, there is no difficulty in developing convergent series. Nevertheless in certain problems, particularly those in which

$$\int (\mathbf{E} \times \mathbf{H}) d\tau$$

does not change as time goes on, the first method is very convenient and less laborious than the second. Obviously the two methods are equivalent, and must lead to exactly the same result.

Whichever method is used, it is necessary to solve the field equations (2) to (5) for **E** and **H**. Lorentz's[1] solution is as follows:

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\dot{\mathbf{A}},$$

$$\mathbf{H} = \nabla \times \mathbf{A},$$

where

$$\phi \equiv \frac{1}{4\pi}\int \frac{[\rho]}{r} d\tau,$$

$$\mathbf{A} \equiv \frac{1}{4\pi c}\int \frac{[\rho \mathbf{v}]}{r} d\tau,$$

the quantities in brackets being retarded, *i. e.*, values of ρ and $\rho\mathbf{v}$ respectively at a time r/c earlier.

For a point charge these reduce to the familiar Lienard[2] potentials

$$\phi \equiv \frac{e}{4\pi\left[r\left(1 - \frac{v_r}{c}\right)\right]},$$

$$\mathbf{A} \equiv \frac{e[\mathbf{v}]}{4\pi c\left[r\left(1 - \frac{v_r}{c}\right)\right]}.$$

Differentiating these retarded potentials, we obtain the usual expressions for **E** and **H** due to a point charge[3] at a time r/c later,

$$\mathbf{E} = \frac{e(1-\beta^2)}{4\pi r^3\left(1 - \frac{v_r}{c}\right)^3}\left\{\left(\mathbf{r} - \frac{r}{c}\mathbf{v}\right) + \frac{\left\{\mathbf{f} \times \left(\mathbf{r} - \frac{r}{c}\mathbf{v}\right)\right\} \times \mathbf{r}}{c^2(1-\beta^2)}\right\} \qquad (11)$$

[1] Theory of Electrons, p. 17 *et seq.*

[2] Eclairage Electrique, 16, p. 5, 1898.

[3] M. Abraham, Theorie der Electrizität, 2, p. 97.

$$\mathbf{H} = \frac{e(1-\beta^2)}{4\pi r^3\left(1-\frac{v_r}{c}\right)^3}\left\{-\frac{1}{c}\mathbf{r}\times\mathbf{v}+\frac{\mathbf{r}\times\left(\left\{\mathbf{f}\times\left(\mathbf{r}-\frac{r}{c}\mathbf{v}\right)\right\}\times\mathbf{r}\right)}{c^2(1-\beta^2)}\right\} \tag{12}$$

where $\beta \equiv v/c$ and $\mathbf{f}$ is the acceleration.

From these it appears that

$$\mathbf{H} = \frac{1}{r}(\mathbf{r}\times\mathbf{E}). \tag{13}$$

Consider now a radiating body, such as a star, which is moving with a velocity V relative to the reference frame to which we apply the electrodynamic equations. The total force due to the emitted radiation will consist of two parts, (*a*) the reaction on each oscillator of the radiation which it, itself, emits, (*b*) the force exerted on each oscillator by the radiation proceeding from the neighboring oscillators. Now to compute the reaction on the aggregate of material oscillators by the rigorous method we are going to pursue would be exceedingly involved. Fortunately we can simplify the problem to the extent of dealing with a single oscillator, *i. e.*, a single vibrating electron, and yet obtain a result that will be a perfectly general test of Larmor's expression for the radiation retardation. For this expression gives the retarding force as a function of the rate of total radiation and the velocity of the radiating body, and of these quantities alone. Hence if the ether exerts a reaction on a group of moving oscillators, it will exert a similar reaction on a single oscillator; and conversely, if there is no reaction on a single vibrating electron due to its drift velocity, there can be none on a group of such vibrators.

Reaction of the Radiation.

Method *A*.

To find the reaction of the radiation, Larmor uses method *A*. The following reasoning is somewhat more rigorous than his, but is substantially the same and leads to the same result, provided the same approximations are made.

Draw a fixed sphere of radius r (Fig. 1) with center at the point occupied by the vibrating electron at a time r/c earlier. Take the X axis in the direction of the velocity which the electron had at this earlier time. Let r be very great compared to the linear dimensions of the electron. Then terms involving r^{-2} will be negligible compared to those in r^{-1}, and $\mathbf{E}$ and $\mathbf{H}$ at the surface of this sphere will be at right angles to the radius vector. Hence

$$\begin{aligned}\mathbf{K}_1 &= -\tfrac{1}{2}\int(E^2+H^2)d\sigma\\ &= -\int u\,d\sigma,\end{aligned}$$

where u is the energy density of the radiation.

Hence, in the X direction

$$K_{1_x} = -\int u \cos\theta \, d\sigma.$$

Now consider the part $\mathbf{K}_2'$ of $\mathbf{K}_2$ due to the fact that the electron's

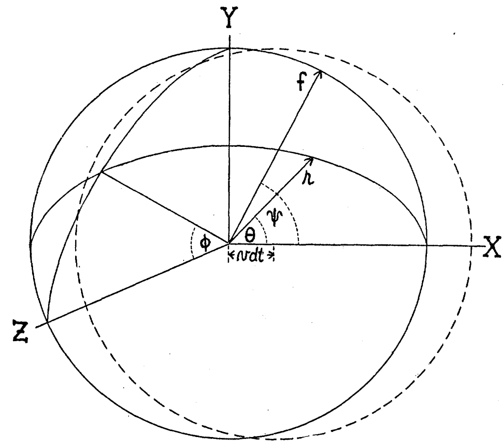

Fig. 1.

field is moving with it. Since the flow of energy at the surface of the sphere is along the radius vector

$$|\mathbf{E} \times \mathbf{H}| = u$$

and, as is obvious from the figure,

$$\frac{d}{dt}\left[\int (\mathbf{E} \times \mathbf{H})_x d\tau\right] = -\int uv \cos^2\theta \, d\sigma$$

or

$$K_{2_x}' = \int u\beta \cos^2\theta \, d\sigma,$$

hence

$$K_x' = -\int u(1 - \beta\cos\theta)\cos\theta \sin\theta \, d\theta \, d\phi \qquad (14)$$

is the force due to the stresses over the surface of the sphere plus that due to the rate of decrease of electromagnetic momentum occasioned by the translation of its field with the electron. That part of $\mathbf{K}_2$ due to the rate of decrease of electromagnetic momentum inside a sphere of radius r moving with the electron is zero when averaged over a whole number of periods, provided the electron's field at the end of this time is the same as it was initially.

From (13)

$$\mathbf{E} \times \mathbf{H} = \frac{1}{r}\left\{ E^2\mathbf{r} - \mathbf{E}\cdot\mathbf{r}\mathbf{E} \right\}.$$

$$= \frac{1}{r} E^2\mathbf{r}.$$

Hence, from (11), (14) becomes

$$K_x' = -\frac{e^2}{16\pi^2c^4}\left\{f^2\int\frac{\cos\theta\sin\theta d\theta d\phi}{(1-\beta\cos\theta)^3} + 2\beta\int\frac{f_r f_v\cos\theta\sin\theta d\theta d\phi}{(1-\beta\cos\theta)^4} - (1-\beta^2)\int\frac{f_r^2\cos\theta\sin\theta d\theta d\phi}{(1-\beta\cos\theta)^5}\right\},$$

where, without loss of generality, we can assume **f** to lie in the XY plane, so that

$$f_v = f\cos\psi,$$

$$f_r = f(\cos\psi\cos\theta + \sin\psi\sin\theta\sin\phi).$$

Performing the integration over the surface of the sphere

$$K_x' = -\frac{e^2\beta f^2}{6\pi c^4(1-\beta^2)^3}\left\{1-\beta^2\sin^2\psi\right\}.$$

Similarly

$$K_y' = -\frac{e^2}{16\pi^2c^4}\left\{f^2\int\frac{\sin^2\theta\sin\phi d\theta d\phi}{(1-\beta\cos\theta)^3} + 2\beta\int\frac{f_r f_v\sin^2\theta\sin\phi d\theta d\phi}{(1-\beta\cos\theta)^4} - (1-\beta^2)\int\frac{f_r^2\sin^2\theta\sin\phi d\theta d\phi}{(1-\beta\cos\theta)^5}\right\},$$

which gives on integration

$$K_y' = 0.$$

From symmetry

$$K_z' = 0.$$

Hence, to the first degree of approximation

$$\mathbf{K}' = -\frac{e^2f^2}{6\pi c^5}\mathbf{v}. \tag{15}$$

Now, to the same degree of approximation, the rate of radiation from the electron is given by

$$R = \frac{e^2f^2}{6\pi c^3}.$$

Hence

$$\mathbf{K}' = -\frac{1}{c^2}R\mathbf{v}.$$

Consider now an electron which is vibrating and at the same time moving in the l direction with a drift velocity V. At any instant

$$K_l' = -\frac{1}{c^2}Rv_l.$$

For a whole number of periods, the impulse is

$$\int K_l'dt = -\frac{1}{c^2}\overline{R}\int v_l dt,$$

where $\overline{R}$ is the mean rate of radiation. But

$$\int v_l dt = Vt.$$

Hence, on the average

$$\overline{K}_l' = -\frac{1}{c^2}\overline{R}V.$$

This is the expression found by Larmor for the resistance due to the reaction of the radiation. But are we justified in neglecting the part of K_{2_l} which depends upon the decrease in the integral

$$\int (\mathbf{E} \times \mathbf{H})_l d\tau$$

taken over the region enclosed by a sphere of radius r moving with the electron? The average impulse due to this part of the total force during a time t is

$$\int K_l'' dt = -\frac{1}{c^2}\left\{\left[\int (\mathbf{E} \times \mathbf{H})_l d\tau\right]_t - \left[\int (\mathbf{E} \times \mathbf{H})_l d\tau\right]_0\right\}.$$

Now the integrals within the brackets are equal, and hence annul each other, if, and only if, the field within the moving sphere of radius r is the same at the end of the whole number of periods over which we are averaging as it was at the beginning, that is to say, if the periodic motion of the electron is undamped. But the energy of a radiating electron is continually decreasing, and consequently its motion cannot be truly periodic unless energy is supplied to it from some outside source. But if energy is to be supplied it must be shown that no impulse on the electronic vibrator accompanies the transfer. The author has not succeeded in devising a method by which a transfer of electromagnetic energy might be effected in such a way that the impulse imparted could be easily calculated. Energy from non-electromagnetic sources—such, for example, as the energy imparted to the radiating electrons on the sun's surface from its gravitational potential energy as the whole mass shrinks—must be excluded from consideration on account of insufficient knowledge of the laws governing the intricate phenomena concerned. In fact, our problem is essentially one in electrodynamics, and the connection between gravitation and electrodynamics is unknown. Consequently in our further treatment of the problem we shall assume that the electron is left to itself and that its radiation is at the expense of the energy of its vibration.

Moreover, from the standpoint of the electron theory, Lorentz[1] has shown that the dynamical equation of an electron contains a damping force which depends upon the rate of change of acceleration, and which is independent of any assumptions as to the distribution of the charge.

[1] Theory of Electrons, p. 49.

In fact it is easily shown that the energy radiated is accounted for by the work done against this resisting force. From this point of view as well, then, an undamped periodic vibration is impossible unless energy is supplied from some outside source.

It may be urged that by making the mass of the electronic vibrator sufficiently large, the diminution in energy due to its radiation and consequently the value of the part of $\mathbf{K}_2$ which we have neglected may be made as small as desired. But it must be remembered that increase in mass involves decrease in the radius of the electron, and hence the volume integral whose decrease we have neglected has to be extended to regions where **E** and **H** are very large, and where any proportionately small change in these quantities will account for a relatively large change in the integral.

Although we are not going to complete the solution of our problem by the method we are here pursuing—for the analytical difficulties in evaluating

$$\int (\mathbf{E} \times \mathbf{H}) d\tau$$

are far more formidable than those encountered in the equivalent method *B*—it may not be superfluous to show the existence of a force which exactly compensates the resistance found by Larmor. [We are dealing

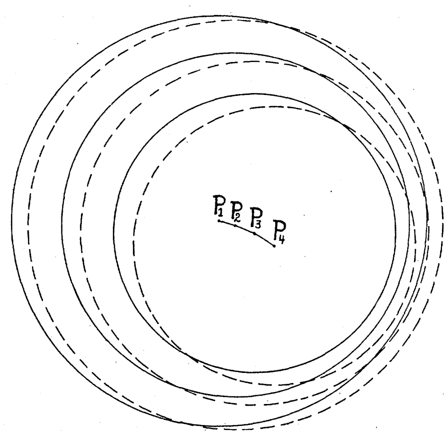

Fig. 2.

here with a single vibrating electron which is receiving no energy from outside sources.] Equation (14) gives $\mathbf{K}'$ for the time 0 in terms of **f** and **v** at a time $-(r/c)$, where r is the radius of the sphere over whose surface the integration is to be performed, r being very large compared to the linear dimensions of the electron. Let P_1 (Fig. 2) be the position of the electron at the time $-(r/c)$, P_2 the position at the time $-(r/c)+dt$,

P_3 that at the time $-(r/c) + 2dt$, etc. Let the outer full-line circle be the trace of a sphere of radius r with center P_1, and the outer dotted circle that of a sphere of the same radius but center P_2. Let the next full-line circle have center P_2 and radius $r - cdt$, and the innermost center P_3 and radius $r - 2cdt$, the dotted circles having respectively the same radii but centers at P_3 and P_4. For the time 0 then, $\mathbf{E}$ and $\mathbf{H}$ over the outer full-line sphere will depend upon the velocity and acceleration which the electron had when at P_1, while for the second full-line sphere the velocity and acceleration of the electron when at P_2 are the ones that must be taken into consideration. At a time dt later, the full-line spheres must be replaced by the dotted spheres, and the effective positions P_2, P_3, and P_4 made use of instead of P_1, P_2, and P_3. Now as the regions between these spheres are far from the electron, the parts of $\mathbf{E}$ and $\mathbf{H}$ having r^{-2} as a factor are negligible compared to those involving only the inverse first power. Hence the flow of energy is along the radius vector, and the value of

$$\int (\mathbf{E} \times \mathbf{H})_x d\tau$$

for the region between the first and second dotted spheres at the time dt will be the same as the value of this integral for the region between the second and third full line spheres at the time 0, and so on. Hence at least part of the decrease in the total integral will be the value of the integral for the region between the two outer full-line spheres. Since the distance between these spheres is

$$cdt(1 - \beta \cos \theta)$$

we find for this part of K_{2_x}

$$K_{2_x}''' = \int u(1 - \beta \cos \theta) \cos \theta \sin \theta d\theta d\phi$$

which exactly annuls the expression (14) *previously obtained.* This is as would be expected, since it is not to be supposed that the reaction on the electron would depend upon the velocity and acceleration which it had at a time r/c previous, where r may be made indefinitely great, but at most upon the state of motion at a time a/c earlier, where a is its greatest linear dimension. The portion of the integral which is conditioned by the state of motion at this comparatively more recent time is that in the vicinity of the electron. On account of the difficulty of developing a convergent series for $\mathbf{E}$ and $\mathbf{H}$ we will not evaluate this integral directly, but resort to the equivalent method B.[1]

[1] On the dynamical theory of the ether as developed in particular by the English school of physicists, the force exerted by radiant energy on matter is conceived to be due to a transfer of momentum from the ether to the body affected. Let us consider the problem under discussion from this point of view. The ether inside the large sphere of radius r (this sphere

METHOD B.

This method consists in obtaining **E** and **H** from the equations of the electrodynamic field, substituting in the expression for the force exerted, and integrating over the region occupied by the electron. In order to carry out the solution we are obliged to make certain assumptions regarding the shape and distribution of charge of the electron. However, we are at liberty to make any such assumptions we choose, for the expression found by Larmor for the radiation reaction is independent of the shape or distribution of charge. As a matter of fact we shall see that terms of the form of Larmor's expression are independent of any such assumptions.

Larmor speaks of the radiation reaction found by him as a *first order* effect. As a term in the equation of motion of the electron it must be considered of the *fifth order*. For if A is the amplitude of the electron's vibration, it is obvious that

$$\frac{fA}{c^2} \text{ is of the order } \beta^2,$$

$$\frac{f^2A^2}{c^4}\beta \text{ is of the order } \beta^5, \text{ and}$$

$$\frac{f^2a^2}{c^4}\beta < \frac{f^2A^2}{c^4}\beta,$$

where a is the radius of the electron. It is this last quantity which is involved in his result. Consequently we shall retain in our analysis all terms of the first five orders. Fortunately a great many complications, such as variations in the distribution of charge on the electron due to its state of motion, do not enter until the sixth order is reached.[1]

Our first step is to expand the retarded expressions (11) and (12) for **E** and **H** due to a point charge in terms of the actual velocity and its derivatives. Suppose we have a charge e at a point whose coördinates are x, y, and z at a time 0, and let $\mathbf{v}$, $\mathbf{f}$, $\dot{\mathbf{f}}$, etc., be its velocity, acceleration,

must be large compared to the diameter of the electron if terms involving r^{-2} are to be neglected as compared to those in r^{-1} but may be very small compared to a millimeter) together with the electron at its center is losing momentum to the ether outside, and since the momentum passing out in the direction of motion is greater than that passing out in the opposite direction, there is a force of exactly the amount found by Larmor. But the ether inside this sphere is also losing momentum in the direction of motion due to the damping of the vibration. Now, by the law of conservation of momentum,

Momentum lost by electron = Momentum gained by ether outside sphere—Momentum lost by ether inside sphere.

Method B will show that the terms on the right-hand side of this equation (the second of which is overlooked by Larmor) must be equal and hence annul each other.

[1] Relativity and the Ether, p. 185.

rate of change of acceleration, etc., at this instant. Now expressing (11) in scalar form, we have for the x component of the electric intensity at the origin at the time 0

$$E_x = \frac{e}{4\pi r_e^2}\left(1 + \frac{\mathbf{v}_e\cdot\mathbf{r}_e}{cr_e}\right)^{-3}\left\{\left(1 - \frac{v_e^2}{c^2} - \frac{\mathbf{f}_e\cdot\mathbf{r}_e}{c^2}\right)\left(-\frac{x_e}{r_e} - \frac{v_{e_x}}{c}\right) - \frac{f_{e_x} r_e}{c^2}\left(1 + \frac{\mathbf{v}_e\cdot\mathbf{r}_e}{cr_e}\right)\right\}, \quad (16)$$

where the quantities with subscript e refer to the effective position of the charge, *i. e.*, its position at a time r_e/c earlier. Hence

$$x_e = x - v_x\frac{r_e}{c} + \frac{1}{2}f_x\frac{r_e^2}{c^2} - \frac{1}{6}\dot{f}_x\frac{r_e^3}{c^3} + \frac{1}{24}\ddot{f}_x\frac{r_e^4}{c^4} - \frac{1}{120}\dddot{f}_x\frac{r_e^5}{c^5}\cdots,$$

$$v_{e_x} = v_x - f_x\frac{r_e}{c} + \frac{1}{2}\dot{f}_x\frac{r_e^2}{c^2} - \frac{1}{6}\ddot{f}_x\frac{r_e^3}{c^3} + \frac{1}{24}\dddot{f}_x\frac{r_e^4}{c^4}\cdots,$$

$$f_{e_x} = f_x - \dot{f}_x\frac{r_e}{c} + \frac{1}{2}\ddot{f}_x\frac{r_e^2}{c^2} - \frac{1}{6}\dddot{f}_x\frac{r_e^3}{c^3}\cdots$$

and similar expressions for the y and z components. Put

$$m_x \equiv \frac{x}{r}, \quad \beta_x \equiv \frac{v_x}{c}, \quad \gamma_x \equiv \frac{f_x r}{c^2}, \quad \delta_x \equiv \frac{\dot{f}_x r^2}{c^3}, \quad \epsilon_x \equiv \frac{\ddot{f}_x r^3}{c^4}, \quad \zeta_x \equiv \frac{\dddot{f}_x r^4}{c^5}.$$

Then

$$r_e^2 = r^2\left\{1 - 2\boldsymbol{\beta}\cdot\mathbf{m}\frac{r_e}{r} + (\boldsymbol{\gamma}\cdot\mathbf{m} + \beta^2)\frac{r_e^2}{r^2} - \frac{1}{3}(\boldsymbol{\delta}\cdot\mathbf{m} + 3\boldsymbol{\gamma}\cdot\boldsymbol{\beta})\frac{r_e^3}{r^3} + \frac{1}{12}(\boldsymbol{\epsilon}\cdot\mathbf{m} + 4\boldsymbol{\delta}\cdot\boldsymbol{\beta} + 3\gamma^2)\frac{r_e^4}{r^4} - \frac{1}{60}(\boldsymbol{\zeta}\cdot\mathbf{m} + 5\boldsymbol{\epsilon}\cdot\boldsymbol{\beta} + 10\boldsymbol{\delta}\cdot\boldsymbol{\gamma})\frac{r_e^5}{r^5}\cdots\right\}.$$

Put

$$k \equiv (1 - \beta^2)^{-\frac{1}{2}},$$

$$\tau \equiv \frac{r_e}{rk},$$

$$a \equiv \boldsymbol{\beta}\cdot\mathbf{m}k,$$

$$b \equiv \boldsymbol{\gamma}\cdot\mathbf{m}k^2,$$

$$c \equiv (\boldsymbol{\delta}\cdot\mathbf{m} + 3\boldsymbol{\gamma}\cdot\boldsymbol{\beta})k^3,$$

$$d \equiv (\boldsymbol{\epsilon}\cdot\mathbf{m} + 4\boldsymbol{\delta}\cdot\boldsymbol{\beta} + 3\gamma^2)k^4,$$

$$e \equiv (\boldsymbol{\zeta}\cdot\mathbf{m} + 5\boldsymbol{\epsilon}\cdot\boldsymbol{\beta} + 10\boldsymbol{\delta}\cdot\boldsymbol{\gamma})k^5,$$

where a is of the first order, b of the second, and so on. Then

$$\tau^2 = 1 - 2a\tau + b\tau^2 - \frac{1}{3}c\tau^3 + \frac{1}{12}d\tau^4 - \frac{1}{60}e\tau^5\cdots, \quad (17)$$

$$1+\frac{\mathbf{v}_e\cdot\mathbf{r}_e}{cr_e}=\frac{1}{k^2\tau^2}\left\{1-a\tau+0+\frac{1}{6}c\tau^3-\frac{1}{12}d\tau^4+\frac{1}{40}e\tau^5\cdots\right\} \tag{18}$$

$$\equiv\frac{I}{k^2\tau^2}.$$

After some reduction we find that

$$\left(1-\frac{v_e^2}{c^2}-\frac{\mathbf{f}_e\cdot\mathbf{r}_e}{c^2}\right)\left(-\frac{x_e}{r_e}-\frac{v_{e_x}}{c}\right)-\frac{f_{e_x}r_e}{c^2}\left(1+\frac{\mathbf{v}_e\cdot\mathbf{r}_e}{cr_e}\right)$$

$$=-\frac{1}{k^3\tau^3}\left\{m_x\left(1-2a\tau+0+\frac{4}{6}c\tau^3-\frac{5}{12}d\tau^4+\frac{6}{40}e\tau^5\cdots\right)\right.$$

$$+0$$

$$-\frac{1}{2}\gamma_xk^2\tau^2\left(-1+0+0+\frac{2}{6}c\tau^3-\frac{3}{12}d\tau^4+\frac{4}{40}e\tau^5\cdots\right)$$

$$+\frac{1}{3}\delta_xk^3\tau^3\left(-2+a\tau+0+\frac{1}{6}c\tau^3-\frac{2}{12}d\tau^4+\frac{3}{40}e\tau^5\cdots\right)$$

$$-\frac{1}{8}\epsilon_xk^4\tau^4\left(-3+2a\tau+0+0-\frac{1}{12}d\tau^4+\frac{2}{40}e\tau^5\cdots\right)$$

$$\left.+\frac{1}{30}\zeta_xk^5\tau^5\left(-4+3a\tau+0-\frac{1}{6}c\tau^3+0+\frac{1}{40}e\tau^5\cdots\right)\cdots\right\} \tag{19}$$

$$\equiv-\frac{J}{k^3\tau^3}.$$

Hence

$$E_x=-\frac{e}{4\pi r^2}k\tau I^{-3}J. \tag{20}$$

Returning to (17) and solving for τ by successive approximations, we find

$$\tau=1-a\left(1-\frac{1}{2}a+0+\frac{1}{8}a^3+0\right)$$

$$+\frac{1}{2}b\left(1-2a+\frac{3}{2}a^2+0\right)+\frac{3}{2}b^2\left(1-\frac{8}{3}a\right)$$

$$-\frac{1}{6}c\,(1-3a+4a^2)-\frac{1}{3}bc$$

$$+\frac{1}{24}d\,(1-4a)$$

$$-\frac{1}{120}e.$$

Substituting this value of τ in (20), we obtain after a laborious reduction

$$E_x = -\frac{ek}{4\pi r^2}\left\{m_x\left(1 - \frac{3}{2}a^2 + \frac{15}{8}a^4\cdots + \frac{1}{2}b - \frac{9}{4}a^2b\cdots + \frac{3}{8}b^2 + \frac{1}{2}ac - \frac{1}{8}d + \frac{1}{15}e\cdots\right) + \frac{1}{2}\gamma_x k^2\left(1 - \frac{3}{2}a^2 + \frac{3}{2}b - \frac{4}{3}c\cdots\right) - \frac{2}{3}\delta_x k^3\left(1 - \frac{3}{2}a + 2b\cdots\right) + \frac{3}{8}\epsilon_x k^4\left(1 - \frac{8}{3}a\cdots\right) - \frac{4}{30}\zeta_x k^5\cdots\right\}. \quad (21)$$

We need **H** to the fourth order only, as it is multiplied by $\mathbf{v}/c$ in the force equation (6). It is obtained most easily from (13), the x component of which is

$$H_x = -\frac{y_e}{r_e}E_z + \frac{z_e}{r_e}E_y,$$

which gives, after considerable reduction

$$H_x = -\frac{ek}{4\pi r^2}\left\{(\beta_y m_z - \beta_z m_y)\left(1 - \frac{3}{2}a^2 + \frac{1}{2}b\cdots\right) + ak(\gamma_y m_z - \gamma_z m_y) - \tfrac{1}{2}k^2(\delta_y m_z - \delta_z m_y) + \tfrac{1}{3}k^3(\epsilon_y m_z - \epsilon_z m_y) - \tfrac{1}{2}k^2(\gamma_y\beta_z - \gamma_z\beta_y) + \tfrac{2}{3}k^3(\delta_y\beta_z - \delta_z\beta_y)\cdots\right\}. \quad (22)$$

If now, we wish the electric and magnetic intensities at a point x, y, z due to a charge at the origin, we must change the signs of the coördinates in (21) and (22), and have

$$E_x = \frac{ek}{4\pi r^2}\left\{m_x\left(1 - \frac{3}{2}a_1^2 + \frac{15}{8}a_1^4\cdots - \frac{1}{2}b_1 + \frac{9}{4}a_1^2b_1\cdots + \frac{3}{8}b_1^2 + \frac{1}{2}a_1c_1 + \frac{1}{8}d_1 - \frac{1}{15}e_1\cdots\right) - \frac{1}{2}\gamma_x k^2\left(1 - \frac{3}{2}a_1^2 - \frac{3}{2}b_1 + \frac{4}{3}c_1\cdots\right) + \frac{2}{3}\delta_x k^3\left(1 + \frac{3}{2}a_1 - 2b_1\cdots\right) - \frac{3}{8}\epsilon_x k^4\left(1 + \frac{8}{3}a_1\cdots\right) + \frac{4}{30}\zeta_x k^5\cdots\right\}, \quad (23)$$

$$H_x = \frac{ek}{4\pi r^2}\left\{(\beta_y m_z - \beta_z m_y)\left(1 - \frac{3}{2}a_1^2 - \frac{1}{2}b_1\cdots\right) - a_1k(\gamma_y m_z - \gamma_z m_y) - \tfrac{1}{2}k^2(\delta_y m_z - \delta_z m_y) + \tfrac{1}{3}k^3(\epsilon_y m_z - \epsilon_z m_y) + \tfrac{1}{2}k^2(\gamma_y\beta_z - \gamma_z\beta_y) - \tfrac{2}{3}k^3(\delta_y\beta_z - \delta_z\beta_y)\cdots\right\}, \quad (24)$$

where

$$a_1 \equiv \boldsymbol{\beta}\cdot\mathbf{m}k,$$
$$b_1 \equiv \boldsymbol{\gamma}\cdot\mathbf{m}k^2,$$
$$c_1 \equiv (\boldsymbol{\delta}\cdot\mathbf{m} - 3\boldsymbol{\gamma}\cdot\boldsymbol{\beta})k^3,$$
$$d_1 \equiv (\boldsymbol{\epsilon}\cdot\mathbf{m} - 4\boldsymbol{\delta}\cdot\boldsymbol{\beta} - 3\gamma^2)k^4,$$
$$e_1 \equiv (\boldsymbol{\zeta}\cdot\mathbf{m} - 5\boldsymbol{\epsilon}\cdot\boldsymbol{\beta} - 10\boldsymbol{\delta}\cdot\boldsymbol{\gamma})k^5.$$

Before we can find the reaction of its field on the electron from these expressions for **E** and **H** we must make some assumption as to the distribution of charge on the electron, just as we should have had to do in method A, if we had attempted to evaluate

$$\int (\mathbf{E} \times \mathbf{H})d\tau$$

in the vicinity of the electron. As already noted, the radiation reaction obtained by Larmor is independent of this distribution, and hence, if existent, must hold irrespective of the assumption we make here.

We might assume the electron to be a rigid conducting sphere—Abraham's electron. The determination of the dynamical equation for such an electron is comparatively simple, and the actual carrying through of the analysis shows the existence of no such resistance as that found by Larmor. However, such an electron is of little interest today, so we shall not burden our readers with the algebra involved. Instead we shall confine ourselves to the deformable electron first proposed by Lorentz, the formula for the mass of which has been abundantly verified experimentally by Bucherer,[1] Neumann,[2] and others. This electron, it will be remembered, contracts when moving, so that its dimensions in the direction of motion are diminished in the ratio of $\sqrt{1 - \beta^2} : 1$. Parenthetically it may be remarked that the Lorentz electron is the only one whose field outside the surface is exactly that of a point charge. We shall take the distribution of charge to be such that the electron, when at rest, is a uniformly charged spherical shell.

At first we shall restrict ourselves to motion in a straight line. Take the X axis as the direction of the velocity. Then since the electron contracts as its velocity increases, the velocity and its derivatives at a time 0 will be less for a point P than for a point O, if P is a distance x farther along the X axis than O. In fact we easily see that after a time dt has elapsed

$$x_t = x\left(1 + \frac{\partial v}{\partial x}dt + \frac{1}{2}\frac{\partial f}{\partial x}dt^2 + \frac{1}{6}\frac{\partial \dot{f}}{\partial x}dt^3\cdots\right).$$

[1] Phys. Zeitschr., 9, p. 755, 1908.

[2] Ann. d. Physik, 45, p. 529, 1914.

But

$$x_t k_t = xk,$$

where

$$v_t = v + f dt + \frac{1}{2}\dot{f}dt^2 + \frac{1}{6}\ddot{f}dt^3 \cdots.$$

Equating coefficients of like powers of dt in these equivalent expressions for x_t, we find

$$\frac{\partial v}{\partial x} = -k^2\beta\frac{f}{c}, \tag{25}$$

$$\frac{\partial f}{\partial x} = -k^4\frac{f^2}{c^2} - k^2\beta\frac{\dot{f}}{c}, \tag{26}$$

$$\frac{\partial \dot{f}}{\partial x} = -3k^6\beta\frac{f^3}{c^3} - 3k^4\frac{f\dot{f}}{c^2} - k^2\beta\frac{\ddot{f}}{c}, \tag{27}$$

and by carrying out the analysis to the second order of x,

$$\frac{\partial^2 v}{\partial x^2} = 2k^6\beta\frac{f^2}{c^3}(1 + \beta^2) + k^4\beta^2\frac{\dot{f}}{c^2}. \tag{28}$$

Now let x' be the X coördinate of a point Q on the electron relative to O when the electron is at rest, and x this distance when it is in motion. Then

$$x' = \int k dx.$$

But

$$\beta = \beta_0 + \frac{\partial\beta_0}{\partial x}x + \frac{1}{2}\frac{\partial^2\beta_0}{\partial x^2}x^2\cdots.$$

Substituting and integrating

$$x' = k_0 x + \frac{1}{2}k_0^3\beta_0\frac{\partial\beta_0}{\partial x}x^2 + \frac{1}{6}\left(k^3\beta_0\frac{\partial^2\beta_0}{\partial x^2} + k^5(1 + 2\beta_0)\left(\frac{\partial\beta_0}{\partial x}\right)^2\right)x^3\cdots. \tag{29}$$

Equations (25) and (28) show that the coefficient of x^3 is of the sixth order and hence negligible. So we have

$$x' = kx - \frac{1}{2}k^5\beta^2\frac{f}{c^2}x^2\cdots.$$

Hence

$$x = \frac{x'}{k} + \frac{1}{2}k^2 a_1^2\gamma_x,$$

$$y = y',$$

$$z = z',$$

$$r^2 = r'^2\frac{1 + a_1^2 b_1}{1 + a_1},$$

$$\frac{m_x k}{r^2} = \frac{m_x'}{r'^2}\left(1 + \frac{3}{2}a_1^2 + \frac{3}{8}a_1^4 - \frac{3}{2}a_1^2 b_1\cdots\right) + \frac{1}{2}\frac{k^3 a_1^2\gamma_x}{r'^2}\cdots,$$

and we find from (23) for the electric intensity at a point x, y, z due to an element of charge de at the origin

$$dE_x = \frac{de}{4\pi r'^2}\left\{ m_x'\left(1 - \frac{1}{2}b_1 + \frac{3}{8}b_1^2 + \frac{1}{2}a_1c_1 + \frac{1}{8}d_1 - \frac{1}{15}e_1\cdots\right)\right.$$
$$- \frac{1}{2}\gamma_x'k^3\left(1 - 2a_1^2 - \frac{3}{2}b_1 + \frac{4}{3}c_1\cdots\right)$$
$$+ \frac{2}{3}\delta_x'k^4\left(1 + \frac{3}{2}a_1 - 2b_1\cdots\right)$$
$$\left. - \frac{3}{8}\epsilon_x'k^5\left(1 + \frac{8}{3}a_1\cdots\right) + \frac{4}{30}\zeta_x'k^6\cdots\right\}, \tag{30}$$

where

$$m_x' \equiv \frac{x'}{r'}, \quad \gamma_x' \equiv \frac{f_x r'}{c^2}, \quad \delta_x' \equiv \frac{\dot{f}_x r'^2}{c^3}, \quad \epsilon_x' \equiv \frac{\ddot{f}_x r'^3}{c^4}, \quad \zeta_x' \equiv \frac{\dddot{f}_x r'^4}{c^5}.$$

To obtain the force exerted on an element of charge de' by the charge de at the origin, in so far as it is due to the electric intensity, we must multiply (30) by de'. Then integrating with respect to de' we find the force exerted on the rest of the electron by de. Finally, integrating with respect to de we obtain the total force in the X direction due to the reaction on the electron of its own field. The magnetic intensity does not come into the problem in the case of linear motion which we are here discussing, since the force due to the magnetic field is always at right angles to the direction of motion.

Hence neglecting terms which must give rise on integration to equal and opposite pairs of forces, (30) reduces to

$$dE_x = \frac{de}{4\pi r'^2}\left\{ -\frac{1}{2}\frac{f_x r'}{c^2}k^3\left(1 + \frac{x'^2}{r'^2} - 4\beta\frac{f_x r'}{c^2}k^3\cdots\right) + \frac{1}{3}\frac{\dot{f}_x r'^2}{c^3}k^4(2 + 0)\right.$$
$$\left. - \frac{1}{8}\frac{\ddot{f}_x r'^3}{c^4}k^5\left(3 - \frac{x'^2}{r'^2}\right) + \frac{1}{30}\frac{\dddot{f}_x r'^4}{c^5}k^6\left(4 - 2\frac{x'^2}{r'^2}\right)\cdots\right\} \tag{31}$$

and the total reaction

$$K_x = \frac{1}{4\pi}\int\int de\,de'\left\{ -\frac{1}{2}\frac{f_x}{c^2 r'}k^3\left(1 + \frac{x'^2}{r'^2}\right) + 2\beta\frac{f_x^2}{c^4}k^6 + \frac{2}{3}\frac{\dot{f}_x}{c^3}k^4\right.$$
$$\left. - \frac{1}{8}\frac{\ddot{f}_x r'}{c^4}k^5\left(3 - \frac{x'^2}{r'^2}\right) + \frac{1}{30}\frac{\dddot{f}_x r'^2}{c^5}k^6\left(4 - 2\frac{x'^2}{r'^2}\right)\cdots\right\},$$

where we do not take into account the variation of $\mathbf{f}$ and $\dot{\mathbf{f}}$ from point to point on the electron, since reference to (26) and (27) shows that the only term of less than sixth order vanishes upon integration.

Since

$$\int\int r'^m dede' = 3\int\int \frac{x'^2}{r'^2} r'^m dede' = \frac{2^{m+1}}{m+2} a^m e^2$$

we get on integration

$$K_x = -\frac{e^2 f_x}{6\pi a c^2(1-\beta^2)^{\frac{3}{2}}} + \frac{e^2 \mathbf{f}\cdot\boldsymbol{\beta} f_x}{2\pi c^4(1-\beta^2)^3}\cdots + \frac{e^2 \dot{f}_x}{6\pi c^3(1-\beta^2)^2} - \frac{e^2 \ddot{f}_x a}{9\pi c^4(1-\beta^2)^{\frac{5}{2}}} + \frac{e^2 \dddot{f}_x a^2}{18\pi c^5(1-\beta^2)^3}\cdots \quad (32)$$

for the x component of the reaction exerted on the electron by its own field, all terms to and including the fifth order having been retained. The coefficient of f_x is the usual expression for the longitudinal mass, and the third term is the damping effect of the radiation. It is obvious from symmetry that the y and z components of the reaction are zero.

Let us now treat the general case of any type of motion. Consider the axes so oriented that the velocity of the point O on the electron is in the X direction and its acceleration in the XY plane at the instant considered. Let P be another point whose coördinates relative to O are $x, y, 0$. Designate by α the angle which the velocity of O makes with the X axis at the end of the time dt. Then

$$\sin\alpha = \frac{f_y dt}{c\beta} \qquad \cos\alpha = 1. \quad (33)$$

Moreover

$$x_t = x + \left(\frac{\partial v_x}{\partial x}x + \frac{\partial v_x}{\partial y}y\right)dt,$$

$$y_t = y + \left(\frac{\partial v_y}{\partial x}x + \frac{\partial v_y}{\partial y}y\right)dt. \quad (34)$$

But

$$(x_t\cos\alpha + y_t\sin\alpha)^2 k_t^2 + (x_t\sin\alpha - y_t\cos\alpha)^2 = x^2k^2 + y^2, \quad (35)$$

where

$$v_{x_t} = v_x + f_x dt,$$

$$v_{y_t} = v_y + f_y dt.$$

Substituting in (35) the values of $\sin\alpha$ and $\cos\alpha$ from (33) and those of x_t and y_t from (34), we get on equating to zero the coefficients of x^2 xy, and y^2

$$\frac{\partial v_x}{\partial x} = -k^2\beta\frac{f_x}{c},$$

$$\frac{\partial v_y}{\partial y} = 0,$$

$$\frac{\partial v_x}{\partial y} + k^{-2}\frac{\partial v_y}{\partial x} + \beta\frac{f_y}{c} = 0.$$

Now, if there is to be no rotation of the electron as a whole

$$\frac{\partial v_x}{\partial y} = 0.$$

Hence

$$\frac{\partial v_x}{\partial x} = -k^2\beta\frac{f_x}{c}, \quad \frac{\partial v_x}{\partial y} = 0,$$

$$\frac{\partial v_y}{\partial x} = -k^2\beta\frac{f_y}{c}, \quad \frac{\partial v_y}{\partial y} = 0. \tag{36}$$

Now consider a point Q near P. Let dx', dy', 0 be the coördinates of Q relative to P when the electron is at rest, and dx, dy, 0 these coördinates when it is in motion. Let dr' and dr respectively denote the distance PQ under the same conditions, let α be the angle which r makes with the X axis, and let θ be the angle which the instantaneous velocity of P makes with this axis. Then

$$dr'^2 = k^2dr^2\cos^2\theta + dr^2\sin^2\theta.$$

But

$$\tan(\alpha - \theta) = \frac{v_y}{v_x} = -k^2\frac{f_y}{c^2}x.$$

Hence

$$dr' = kdr\sqrt{1 - \beta^2\sin^2\theta} - k^5dx\beta^2\frac{f_x}{c^2}x\cos\theta - k^3dx\beta^2\frac{f_y}{c^2}x\sin\theta.$$

If $\theta = 0$

$$dx' = kdx - k^5\beta^2\frac{f_x}{c^2}xdx,$$

$$x' = kx - \frac{1}{2}k^5\beta^2\frac{f_x}{c^2}x^2.$$

If $\theta = 90°$

$$dy' = dy - k^3\beta^2\frac{f_y}{c^2}xdx,$$

$$y' = y - \frac{1}{2}k^3\beta^2\frac{f_y}{c^2}x^2.$$

Hence

$$x = \frac{x'}{k} + \frac{1}{2}k^2a_1^2\gamma_x,$$

$$y = y' + \frac{1}{2}a_1^2\gamma_y,$$

$$r^2 = r'^2\frac{1 + a_1^2b_1}{1 + a_1^2},$$

$$\frac{m_xk}{r^2} = \frac{m_x'}{r'^2}\left(1 + \frac{3}{2}a_1^2 + \frac{3}{8}a_1^4 - \frac{3}{2}a_1^2b_1\cdots\right) + \frac{1}{2}\frac{k^3a_1^2\gamma_x}{r'^2}\cdots,$$

$$\frac{m_yk}{r^2} = \frac{m_y'k}{r'^2}\left(1 + \frac{3}{2}a_1^2 + \frac{3}{8}a_1^4 - \frac{3}{2}a_1^2b_1\cdots\right) + \frac{1}{2}\frac{ka_1^2\gamma_y}{r'^2}\cdots. \tag{37}$$

Therefore we obtain from (23) for the x component of the electric intensity exactly the same expression (30) as in the case of linear motion. So far as the part of the x component of the force which depends upon the magnetic field is concerned, the first term which does not vanish on integration is of the sixth order and hence negligible. Remembering that

$$\int\int \frac{x'}{r'} \frac{y'}{r'} r'^m dede' = 0$$

and integrating, we get for the x component of the reaction exerted on the electron by its field the same expression (32) as in the case of linear motion.

For the Y direction we obtain from (23) and (37)

$$\begin{aligned} dE_y = \frac{de}{4\pi r'^2} \Big\{ & m_y'k\left(1 - \frac{1}{2}b_1 + \frac{3}{8}b_1^2 + \frac{1}{2}a_1c_1 + \frac{1}{8}d_1 - \frac{1}{15}e_1 \cdots\right) \\ & - \frac{1}{2}\gamma_y'k^3\left(1 - 2a_1^2 - \frac{3}{2}b_1 + \frac{4}{3}c_1 \cdots\right) \\ & + \frac{2}{3}\delta_y'k^4\left(1 + \frac{3}{2}a_1 - 2b_1 \cdots\right) \\ & - \frac{3}{8}\epsilon_y'k^5\left(1 + \frac{8}{3}a_1 \cdots\right) + \frac{4}{30}\zeta_y'k^6 \cdots \Big\}. \end{aligned}$$

Neglecting terms which give rise to equal and opposite pairs of forces, this reduces to

$$\begin{aligned} dE_y = \frac{de}{4\pi r'^2} \Big\{ & -\frac{1}{2}\frac{f_y r'}{c^2}k^3\left(1 + \frac{y'^2}{r'^2} - 2\beta^2\frac{x'^2}{r'^2} - 4\beta\frac{f_x r'}{c^2}k^3 \cdots\right) \\ & + \frac{1}{3}\frac{\dot{f}_y r'^2}{c^3}k^4(2 + 0) - \frac{1}{8}\frac{\ddot{f}_y r'^3}{c^4}k^5\left(3 - \frac{y'^2}{r'^2}\right) \\ & + \frac{1}{30}\frac{\dddot{f}_y r'^4}{c^5}k^6\left(4 - 2\frac{y'^2}{r'^2}\right)\cdots \Big\}. \end{aligned} \tag{38}$$

Also, from (24) and (37)

$$\begin{aligned} \frac{1}{c}(\mathbf{v} \times d\mathbf{H})_y &= -\beta dH_z \\ &= \frac{kde}{4\pi r'^2}\{-\beta^2 m_y(1 - \tfrac{1}{2}b_1 \cdots) + a_1\beta k(\gamma_x m_y - \gamma_y m_x) \\ &\quad + \tfrac{1}{2}\beta k^2(\delta_x m_y - \delta_y m_x) - \tfrac{1}{3}\beta k^3(\epsilon_x m_y - \epsilon_y m_x) \\ &\quad + \tfrac{1}{2}\gamma_y\beta^2 k^2 - \tfrac{2}{3}\delta_y\beta^2 k^3 \cdots\}, \end{aligned}$$

which reduces, when we neglect terms which give rise to equal and opposite pairs of forces, to

$$\frac{1}{c}(\mathbf{v}\times d\mathbf{H})_y=\frac{de}{4\pi r'^2}\left\{\frac{1}{2}\frac{f_y r'}{c^2}\beta^2k^3\left(1+\frac{y'^2}{r'^2}-2\frac{x'^2}{r'^2}\cdots\right)\right.$$

$$\left.-\frac{2}{3}\frac{\dot{f}_y r'^2}{c^3}\beta^2k^4\cdots\right\}. \quad (39)$$

Hence

$$dE_y+\frac{1}{c}(\mathbf{v}\times d\mathbf{H})_y=\frac{de}{4\pi r'^2}\left\{-\frac{1}{2}\frac{f_y r'}{c^2}k\left(1+\frac{y'^2}{r'^2}-4\beta\frac{f_x r'}{c^2}k^3\cdots\right)\right.$$

$$+\frac{1}{3}\frac{\dot{f}_y r'^2}{c^3}k^2(2+0)-\frac{1}{8}\frac{\ddot{f}_y r'^3}{c^4}k^3\left(3-\frac{y'^2}{r'^2}\right)$$

$$\left.+\frac{1}{30}\frac{\dddot{f}_y r'^4}{c^5}k^4\left(4-2\frac{y'^2}{r'^2}\right)\cdots\right\} \quad (40)$$

and the total reaction

$$K_y=\frac{1}{4\pi}\int\int dede'\left\{-\frac{1}{2}\frac{f_y}{c^2r'}k\left(1+\frac{y'^2}{r'^2}\right)+2\beta\frac{f_xf_y}{c^4}k^4\right.$$

$$\left.+\frac{2}{3}\frac{\dot{f}_y}{c^3}k^2-\frac{1}{8}\frac{\ddot{f}_y r'}{c^4}k^3\left(3-\frac{y'^2}{r'^2}\right)+\frac{1}{30}\frac{\dddot{f}_y r'^2}{c^5}k^4\left(4-2\frac{y'^2}{r'^2}\right)\cdots\right\}.$$

Integrating we find

$$K_y=-\frac{e^2f_y}{6\pi ac^2(1-\beta^2)^{\frac{1}{2}}}+\frac{e^2\mathbf{f}\cdot\boldsymbol{\beta}f_y}{2\pi c^4(1-\beta^2)^2}\cdots+\frac{e^2\dot{f}_y}{6\pi c^3(1-\beta^2)}$$

$$-\frac{e^2\ddot{f}_ya}{9\pi c^4(1-\beta^2)^{\frac{3}{2}}}+\frac{e^2\dddot{f}_ya^2}{18\pi c^5(1-\beta^2)^2}\cdots, \quad (41)$$

where the coefficient of f_y is the usual transverse mass. We obtain a similar expression for K_z. It is to be noted that the coefficient of $\mathbf{f}\cdot\boldsymbol{\beta}\mathbf{f}$ as well as that of $\dot{\mathbf{f}}$ is *independent of the assumption as to the distribution of the charge.*[1]

It may be of interest to give, in passing, the equation of motion of the deformable electron to all orders—neglecting products of derivatives of the velocity—for the instant when the electron is at rest relative to the observer. The analysis is omitted. We find for the reaction of the electron's field

[1] It is to be noted that in (32) and (41) are obtained for the first time general expressions for the longitudinal and transverse masses respectively which are not limited to a quasi-stationary state of motion.

$$\mathbf{K} = -\frac{e^2\mathbf{f}}{6\pi ac^2} + \frac{e^2\dot{\mathbf{f}}}{6\pi c^3} - \frac{e^2\ddot{\mathbf{f}}a}{9\pi c^4} + \frac{e^2\dddot{\mathbf{f}}a^2}{18\pi c^5} - \frac{e^2\ddddot{\mathbf{f}}a^3}{45\pi c^6} + \frac{e^2\overset{\cdot\cdot\cdot\cdot\cdot}{\mathbf{f}}a^4}{135\pi c^7} - \cdots$$

$$= \frac{e^2}{12\pi a^2 c}\sum_1^\infty \frac{(-1)^h}{h!}\left(\frac{2a}{c}\right)^h \frac{d^h\mathbf{v}}{dt^h}$$

$$= \frac{e^2}{12\pi a^2 c} e^{-\frac{2a}{c}\frac{d}{dt}}\mathbf{v}$$

$$= \frac{e^2}{12\pi a^2 c}\mathbf{v}_{t=-\frac{2a}{c}}. \tag{42}$$

The force exerted on an electron by its own field is equal to a constant multiplied by the velocity which it had at a time earlier equal to the time taken by light to travel across the electron's diameter. Now, if we choose the proper point inside the electron to take as the one to which the derivatives of $\mathbf{v}$ in the equation above apply, we can make the product terms which we have neglected vanish exactly. *So there is a point inside the electron for which* (42) *is the exact equation of motion.*

To return to our problem. Inspection of expressions (32) and (41) shows that the reaction on the electron due to its own field contains terms having the directions (except for the aberration due to the differing powers of $1 - \beta^2$ in the denominators of the components) of $\mathbf{f}$, $\dot{\mathbf{f}}$ and higher derivatives of the acceleration. *There is no term representing a force opposed to the velocity,* as Larmor's result would imply. Hence the reaction constitutes a resistance to the acceleration, etc., and *not* to the velocity of the vibrating electron. *In fact we shall now show that in every term—mass reaction as well as radiation reaction—the form of equations* (32) *and* (41) *is precisely that demanded by the principle of relativity.*

Let symbols without primes refer to a system K (0), which we may for convenience call the rest system, and let symbols with primes refer to a system K (v) which has a velocity v in the X direction relative to K (0). Consider a moving point. Its velocity, acceleration, and higher derivatives relative to an observer in K (v) are found in terms of these quantities relative to an observer in K (0) by differentiating the Lorentz-Einstein transformations. Suppose now that the point is, at the instant considered, at rest in K (v). Then the transformations obtained reduce to

$$f_x' = k^3 f_x,$$

$$f_y' = k^2 f_y,$$

$$\dot{f}_x' = k^4\dot{f}_x + 3k^6\mathbf{f}\cdot\boldsymbol{\beta}\frac{f_x}{c},$$

$$\dot{f}_y' = k^3\dot{f}_y + 3k^5\mathbf{f}\cdot\boldsymbol{\beta}\frac{f_y}{c},$$

$$\ddot{f}_x' = k^5\ddot{f}_x + \text{terms of sixth and higher orders},$$

$$\ddot{f}_y' = k^4\ddot{f}_x + \cdots,$$

$$\dddot{f}_x' = k^6\dddot{f}_x + \cdots,$$

$$\dddot{f}_y' = k^5\dddot{f}_y + \cdots.$$

If now, our point represents an electron, its equation of motion relative to an observer in K (v) is obtained by putting $\beta = 0$ in (32) and (41),

$$eE_x' = \frac{e^2 f_x'}{6\pi a c^2} - \frac{e^2 \dot{f}_x'}{6\pi c^3} + \frac{e^2 \ddot{f}_x' a}{9\pi c^4} - \frac{e^2 \dddot{f}_x' a^2}{18\pi c^5} + \text{higher orders},$$

$$eE_y' = \frac{e^2 f_y'}{6\pi a c^2} - \frac{e^2 \dot{f}_y'}{6\pi c^3} + \frac{e^2 \ddot{f}_y' a}{9\pi c^4} - \frac{e^2 \dddot{f}_y' a^2}{18\pi c^5} + \cdots.$$

But the relativity theory gives the familiar relations

$$E_x' = E_x,$$

$$E_y' = k\left\{E_y + \frac{1}{c}(\mathbf{v} \times \mathbf{H})_y\right\}.$$

Hence we have

$$eE_x = \frac{e^2 f_x}{6\pi a c^2(1-\beta^2)^{\frac{3}{2}}} - \frac{e^2 \mathbf{f}\cdot\boldsymbol{\beta} f_x}{2\pi c^4(1-\beta^2)^3} \cdots - \frac{e^2 \dot{f}_x}{6\pi c^3(1-\beta^2)^2}$$

$$+ \frac{e^2 \ddot{f}_x a}{9\pi c^4(1-\beta^2)^{\frac{5}{2}}} - \frac{e^2 \dddot{f}_x a^2}{18\pi c^5(1-\beta^2)^3} \cdots,$$

$$e\left\{E_y + \frac{1}{c}(\mathbf{v}\times\mathbf{H})_y\right\} = \frac{e^2 f_y}{6\pi a c^2(1-\beta^2)^{\frac{1}{2}}} - \frac{e^2 \mathbf{f}\cdot\boldsymbol{\beta} f_y}{2\pi c^4(1-\beta^2)^2} \cdots$$

$$- \frac{e^2 \dot{f}_y}{6\pi c^3(1-\beta^2)} + \frac{e^2 \ddot{f}_y a}{9\pi c^4(1-\beta^2)^{\frac{3}{2}}} - \frac{e^2 \dddot{f}_y a^2}{18\pi c^5(1-\beta^2)^2} \cdots,$$

which agree exactly with (32) and (41), showing that β enters into the equation of motion of a moving electron in exactly the same way whether we obtain that equation directly from electrodynamics, or obtain it by applying the electrodynamic equations to an electron at rest and then using the kinematical transformations of relativity to find it relative to an observer with respect to whom the electron is in motion. *So we conclude that the equation of motion of an electron as determined from the electrodynamic equations is completely in accord with the principle of relativity, at least as far as the fifth order. Hence a moving vibrator experiences no retardation on account of its radiation.* And since the retardation in question depends only upon the drift velocity and rate of radiation, *this conclusion is equally true of any moving body, however complex.*

SUMMARY.

(*a*) Professor Larmor's deduction from the electrodynamic equations of a radiation reaction on a moving mass has been shown to rest upon a tacit assumption which utterly invalidates his conclusion.

(*b*) It has been shown rigorously that classical electrodynamics leads to no retardation on a moving and radiating mass, but is completely in accord with the principle of relativity.

(*c*) The equation of motion of the Lorentz deformable electron has been computed from the electrodynamic equations as far as and including terms of the fifth order, and found to be in exact agreement with the principle of relativity. The result obtained is more general than any previously published in that it is limited to no particular type of motion, such as quasi-stationary motion in a straight line.

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